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On local uniqueness of weak solutions of the Navier–Stokes system with bounded initial data

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Abstract

We address local uniqueness of weak solutions of the Navier–Stokes system with initial data in $L^\infty(\mathbb{R}^n)$. We show that uniqueness of solutions does not hold. We classify all solutions as all those which can be obtained from the solution of the integral equation (mild solution) by means of a certain transformation.

MSC: 35Q30; 76D05; 35K15

Keywords: Navier–Stokes equations; Weak solutions; Mild solutions; Nonuniqueness

1. Introduction

In this paper, we address uniqueness of solutions of the Navier–Stokes system

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\ \nabla \cdot u &= 0 \end{aligned} \tag{1.1}$$

with the initial datum

$$u(x, 0) = u_0(x), \tag{1.2}$$

which belongs to $L^\infty(\mathbb{R}^n)$.

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Fabes, Jones, and Riviere proved in [FJR] existence and uniqueness of local mild solutions of the Navier–Stokes system if $n<p\leqslant\infty$ ($p=\infty$ included). They showed that the integral form of (1.1)–(1.2) is

$$u(\cdot,t)=\int_0^t k*(u\otimes u)\,ds+e^{t\Delta}u_0$$

(where $e^{t\Delta}u_0$ denotes the solution of the heat equation with the initial datum u_0) with a kernel k which satisfies

$$|k(x,t)|\leqslant\frac{C(n)}{(|x|+t^{1/2})^{n+1}},\quad (x,t)\in\mathbb{R}^n\times(0,\infty).$$

Because of this fact, the mild solution may be constructed using a fixed point argument in the space $L^\infty(0,T,L^p(\mathbb{R}^n))$ for a suitable $T>0$ (as it was done for the nonlinear heat equation in [W]). On the other hand, one also wishes to consider weak solutions when $u_0\in L^p(\mathbb{R}^n)$. This was done in [FJR] by establishing that every solution $u\in L^q(0,T,L^p(\mathbb{R}^n))$ is weak if and only if it is mild—provided $2\leqslant q\leqslant\infty$ and $2\leqslant p<\infty$ ($p=\infty$ is not included). (Koch and Tataru have recently established local well posedness for mild solutions with much more general initial data in BMO_R^{-1}, cf. [KT]; Kato considered in [K] the case $p=n$.)

The present note covers the case $p=\infty$. As it turns out, $p=\infty$ is indeed different from the case $p<\infty$ in the sense that uniqueness of weak solutions does not hold. Namely, $u_1(x,t)=(t,0,\dots,0)$ and $u_2(x,t)=(0,0,\dots,0)$, for $t\geqslant0$, are both weak solutions of the Navier–Stokes system with the initial datum $u_0=(0,0,\dots,0)$. Moreover, every initial datum leads to infinite number of weak solutions: If $u(x,t)$ is a solution, so is its transgalilean transform $u(x-\Phi(t),t)+\phi(t)$ where ϕ is sufficiently regular with $\phi(0)=0$ and where $\Phi(t)=\int_0^t\phi(s)\,ds$ ([FMRT, p. 31]). Our main result is that all solutions can be obtained in this manner. Namely, there exists a solution which may be constructed (essentially) as in [FJR]; every weak solution of the initial value problem can then be obtained from this solution by means of the transformation described above.

The paper is organized as follows. Section 2 contains a definition of weak solutions of the initial value problem and the main result. Section 3 recalls some well-known facts about the space BMO. Then we briefly sketch the proof of existence of mild solutions. Lemma 3.3 is the main step in the proof of the theorem, which is then given in the second part of Section 3.

2. The main theorem

Consider the Navier–Stokes system

$$\partial_t u_k-\Delta u_k+\partial_j(u_ju_k)+\partial_k\pi=0,\quad k=1,\dots,n,$$

$$\partial_j u_j=0. \tag{2.1}$$

(The summation convention on repeated indices is used throughout.) It is straight-forward to consider the case of non-zero forcing as well. The initial condition is

$$u(x,0) = u_0(x) = (u_{01}(x), \dots, u_{0n}(x)), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $u_0 \in L^\infty(\mathbb{R}^n)$ and $\nabla \cdot u_0 = 0$ holds in $\mathscr{D}'(\mathbb{R}^n)$.

Definition 2.1. We say that u is a weak solution of the initial value problem (2.1)–(2.2) on $[0,T)$ if

(i) $u \in L^\infty_{\mathrm{loc}}(0,T,L^\infty(\mathbb{R}^n))$;
(ii) we have

$$\int\int_{S_T} u_k(\partial_t \psi_k + \Delta\psi_k + u_j \partial_j \psi_k) = 0$$

for all

$$\psi = (\psi_1, \dots, \psi_n) \in \{\theta \in C_0^\infty(S_T, \mathbb{R}^n)\colon \ \nabla \cdot \theta = 0\}$$

where $S_T = \mathbb{R}^n \times (0,T)$;
(iii) the initial condition is satisfied in the following sense: The function $t \mapsto u(\cdot,t) - e^{t\Delta}u_0$ is continuous at $t=0$ (as an $L^\infty(\mathbb{R}^n)$-valued function) and its value at 0 is 0.

We shall refer to the weak solutions defined above simply as *solutions of the initial value problem*.

It will turn out that instead of (iii) we may assume equivalently

(iii)$'$ $u(\cdot,t) \to u_0$ in $\mathscr{D}'$ as $t \to 0+$

(cf. Remark 3.5 below).

Theorem 2.2. *There exists at least one solution u of the initial value problem on* $[0,T)$ *where* $T \geqslant 1/C\|u_0\|^2_{L^\infty(\mathbb{R}^n)}$. *Also,* $\tilde{u}$ *is a solution of the initial value problem on* $[0,T)$ *if and only if there exists a function* $\phi \in L^\infty_{\mathrm{loc}}([0,T),\mathbb{R}^n)$ *with* $\lim_{t\to 0+}\phi(t) = \phi(0) = 0$ *such that*

$$\tilde{u}(x,t) = u(x - \Phi(t), t) + \phi(t), \ \ a.e. (x,t) \in S_T$$

where $\Phi(t) = \int_0^t \phi(s)\,ds$.

Here and in the sequel, C denotes a generic positive constant depending only on n. The theorem is proven in Section 3.

3. Proof of the main theorem

For any distributional solution of (2.1), the pressure π satisfies the equation $-\Delta\pi = \partial_{ij}(u_i u_j)$. If $i, j \in \{1, \ldots, n\}$ and $f \in L^p(\mathbb{R}^n)$, where $p \in (1, \infty)$, the equation

$$-\Delta\pi = \partial_{ij} f \tag{3.1}$$

has a unique solution $\pi \in L^p(\mathbb{R}^n)$, which is independent of p. The formula for π is

$$\pi(x) = -\frac{\delta_{ij}}{n} f + \text{P.V.} \int K_{ij}(y) f(x-y)\, dy, \tag{3.2}$$

where

$$K_{ij}(x) = \frac{x_{ij} - \frac{\delta_{ij}}{n}|x|^2}{\omega_n |x|^{n+1}}, \quad x \in \mathbb{R}^n \backslash \{0\},$$

where $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ is the volume of the unit ball in $\mathbb{R}^n$. Moreover, the Calderón–Zygmund theorem states

$$||\pi||_{L^p(\mathbb{R}^n)} \leqslant \frac{Cp^2}{p-1} ||f||_{L^p(\mathbb{R}^n)}, \quad p \in (1, \infty)$$

(cf. [St1]). If $f \in L^\infty(\mathbb{R}^n)$, then Eq. (3.1) has a unique solution π which belongs to

$$\text{BMO} = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_B \frac{1}{|B|} \int_B \left| v(x) - \frac{1}{|B|} \int_B v \right| dx = ||v||_{\text{BMO}} < \infty \right\},$$

where the supremum is taken over all balls in $\mathbb{R}^n$; the functions differing by a constant almost everywhere are considered the same. Formula (3.2) has to be modified slightly (cf. [F]) in such a way that it agrees (modulo a constant) if $f \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for some $p \in (1, \infty)$. Also, the estimate

$$||\pi||_{\text{BMO}} \leqslant C ||f||_{L^\infty(\mathbb{R}^n)}$$

holds. The solution π constructed above will be denoted simply by

$$\pi = K_{ij} f = \partial_i (-\Delta)^{-1} \partial_j f.$$

Also, we recall a fact concerning the problem

$$\partial_t u - \Delta u = \nabla \cdot f, \tag{3.3}$$

where $f = (f_1, \ldots, f_n) \in L^\infty(0, T, \text{BMO})$.

Lemma 3.1. *Let $T > 0$ and $f_j \in L^\infty(0, T, \mathrm{BMO})$ for $j = 1, \dots, n$. Then the integral*

$$\int\int_{S_T} |\nabla G(x-y, t-s)|\,|f(y,s)|\,dy\,ds$$

is finite for every $(x,t) \in S_T = \mathbb{R}^n \times (0,T)$. The function

$$u(x,t) = \int_0^t \int \partial_j G(x-y, t-s) f_j(y,s)\,dy\,ds$$

satisfies $u \in C([0,T), L^\infty(\mathbb{R}^n))$ with

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^{1/2}\|f\|_{L^\infty(0,T,\mathrm{BMO})}.$$

The function u solves (3.3) *in $\mathscr{D}'(S_T)$.*

This lemma is a simple consequence of the inequality

$$|\nabla G(x,t)| \leqslant \frac{C}{(|x| + t^{1/2})^{n+1}}, \quad (x,t) \in \mathbb{R}^n \times (0, \infty)$$

and the estimate

$$\int_{\mathbb{R}^n} \frac{|f(x) - \frac{1}{|B_1|}\int_{B_1} f|}{|x|^{n+1} + 1}\,dx \leqslant C\|f\|_{\mathrm{BMO}}$$

where B_1 is the ball centered at 0 with radius 1 (cf. [S, Lemma 3.9] or [St2, p. 141]).

The existence of mild (integral) solutions is known. For completeness sake, we state the result and sketch the iteration argument.

Let $u_0 \in L^\infty(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Then u is a mild solution of the Navier–Stokes initial value problem if $u \in C((0,T), L^\infty(\mathbb{R}^n))$ and if

$$\begin{aligned} u_k(x,t) = &-\int_0^t \int \partial_j G(x-y, t-s) u_j(y,s) u_k(y,s)\,dy\,ds \\ &-\int_0^t \int \partial_k G(x-y, t-s)\pi(y,s)\,dy\,ds \\ &+\int G(x-y,t) u_{0k}(y)\,dy, \end{aligned} \tag{3.4}$$

where

$$\pi = K_{ij}(u_i u_j) \tag{3.5}$$

holds for every $(x,t) \in S_T = \mathbb{R}^n \times (0,T)$. Denote $M_0 = \|u_0\|_{L^\infty}$.

Proposition 3.2. *There exists a mild solution u of the Navier–Stokes system on* $(0, T)$ *where*

$$T \geqslant \frac{1}{CM_0^2}.$$

This solution is unique within the class of mild solutions and continuity with respect to initial data holds.

The above states that the Navier–Stokes initial value problem is locally well posed within the class of mild solutions. This proposition is proven in [FJR]; however, for completeness sake, we include the proof of existence.

Proof (sketch). As in [GK], let

$$\begin{aligned} u_k^{(m+1)}(x,t) = & -\int_0^t \int \partial_j G(x-y,t-s) u_j^{(m)}(y,s) u_k^{(m)}(y,s)\, dy\, ds \\ & -\int_0^t \int \partial_k G(x-y,t-s) \pi^{(m)}(y,s)\, dy\, ds \\ & + \int G(x-y,t) u_{0k}(y)\, dy \end{aligned}$$

for $k = 1, \dots, n$ with the first two terms on the right-hand side removed if $m = 0$. The pressure is computed from

$$\pi^{(m)} = K_{ij}(u_i^{(m)} u_j^{(m)})$$

for $m = 1, 2, \dots$. By Lemma 3.1, we have

$$u^{(m)} - e^{t\Delta} u_0 \in C([0, T), L^\infty(\mathbb{R}^n))$$

with value 0 at $t = 0$, and $\pi^{(m)} \in C_{\mathrm{b}}((0, T), \mathrm{BMO})$ for all $T > 0$. Moreover,

$$||u^{(m+1)}||_{L^\infty(S_t)} \leqslant Ct^{1/2} ||u^{(m)}||^2_{L^\infty(S_t)} + C_0 M_0, \quad t > 0$$

for $m \in \mathbb{N}$ and

$$||u^{(1)}||_{L^\infty(S_t)} \leqslant C_0 M_0, \quad t > 0$$

for some fixed constant $C_0 = C_0(n)$. Using induction, we get

$$||u^{(m)}||_{L^\infty(S_T)} \leqslant 2C_0 M_0$$

$m \in \mathbb{N}$ provided

$$0 < T \leqslant \frac{1}{CM_0^2} \tag{3.6}$$

with C sufficiently large. Similarly,

$$||u^{(m+1)} - u^{(m)}||_{L^\infty(S_T)} \leqslant \frac{1}{2}||u^{(m)} - u^{(m-1)}||_{L^\infty(S_T)}$$

holds for $m = 2, 3, \dots$ provided (3.6) holds with a suitable C. A fixed point argument then concludes the proof. $\square$

Next, we proceed toward establishing a relationship between solutions of the initial value problem and mild solutions. The following lemma is the main lemma needed in the proof of Theorem 2.2.

Lemma 3.3. *Assume that u is a solution of the initial value problem* (*in the sense of Definition* 2.1) *on* $(0, T)$. *Then there exists a function*

$$\phi = (\phi_1, \dots, \phi_n) \in L^\infty_{\mathrm{loc}}([0, T), \mathbb{R}^n)$$

with $\lim_{t \to 0+} \phi(t) = \phi(0) = 0$ *such that the equation*

$$\partial_t u_k - \Delta u_k + \partial_j(u_j u_k) + \partial_k \pi_0 + {\phi_k}'(t) = 0, \tag{3.7}$$

where $\pi_0 = K_{ij}(u_i u_j)$, *holds in* $\mathscr{D}'(S_T)$ *for all* $k = 1, \dots, n$.

Proof. By de Rham's theorem [CF,T], there exists $\pi \in \mathscr{D}'(S_T)$ such that the first equation in (2.1) holds in $\mathscr{D}'(S_T)$. Denote $\pi_0 = K_{ij}(u_i u_j)$ and $\pi_h = \pi - \pi_0$. Note that $\Delta \pi_h = 0$ in $\mathscr{D}'(S_T)$ by taking the divergence of the first equation in (2.1). First, we claim that

$$\partial_{lk} \pi_h = 0, \quad l, k = 1, \dots, n. \tag{3.8}$$

Let $\alpha \in \mathscr{D}(0, T)$ and $\beta \in \mathscr{D}(\mathbb{R}^n)$ be test functions such that β is radial and $\int \beta = 1$. Also, let

$$\beta_R(x) = \frac{1}{R^n} \beta\left(\frac{x}{R}\right), \quad x \in \mathbb{R}^n$$

for $R > 0$. Then, testing the first equation in (2.1) with $\psi(x) = \alpha(t) \nabla \beta_R(x)$, we get

$$\begin{aligned} \langle \partial_{kl} \pi_h, \alpha(t) \beta_R(x) \rangle = &- \langle u_k, \alpha'(t) \partial_l \beta_R(x) \rangle - \langle u_k, \alpha(t) \Delta \partial_l \beta_R(x) \rangle \\ &- \langle u_j u_k, \alpha(t) \partial_{lj} \beta_R(x) \rangle - \langle \pi_0, \alpha(t) \partial_{kl} \beta_R(x) \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathscr{D}'(S_T)$-$\mathscr{D}(S_T)$ duality. By the mean value theorem and since $\Delta(\alpha(t)\partial_{kl}\pi_h) = 0$ in $\mathscr{D}'(S_T)$, we get

$$\langle \partial_{kl}\pi_h, \alpha(t)\beta_R(x) \rangle = \langle \partial_{kl}\pi_h, \alpha(t)\beta(x) \rangle.$$

Therefore,

$$\begin{aligned}
&\langle \partial_{kl}\pi_h, \alpha(t)\beta(x) \rangle \\
&\quad = -\frac{1}{R^{n+1}}\langle u_k, \alpha'(t)\partial_l\beta(x/R) \rangle - \frac{1}{R^{n+3}}\langle u_k, \alpha(t)\Delta\partial_l\beta(x/R) \rangle \\
&\qquad - \frac{1}{R^{n+2}}\langle u_j u_k, \alpha(t)\partial_{lj}\beta(x/R) \rangle - \frac{1}{R^{n+2}}\langle \pi_0, \partial_{kl}\beta(x/R) \rangle.
\end{aligned}$$

The right-hand side of the above equality converges to 0 as $R \to \infty$, and we obtain $\langle \partial_{kl}\pi_h, \alpha(t)\beta(x) \rangle = 0$ for all $\alpha \in \mathscr{D}(0,T)$ and radial $\beta \in \mathscr{D}(\mathbb{R}^n)$ such that $\int \beta = 1$. Translating in the x-variable, this implies $\langle \partial_{kl}\pi_h, \alpha(t)\beta(x-a) \rangle = 0$ for all α and β as above and all $a \in \mathbb{R}^n$. It is now elementary to deduce (3.8). We conclude that, for every $k \in \{1, \dots, n\}$, $\partial_k\pi_h$ is a distribution which depends only on t; more precisely, there exists $\tilde{\phi}_k \in \mathscr{D}'(0,T)$ such that

$$\langle \partial_k\pi_h, \alpha \rangle = \langle \tilde{\phi}_k, \int \alpha(y, \cdot)\, dy \rangle$$

for all $\alpha \in \mathscr{D}(S_T)$ where the brackets on the right-hand side denote the $\mathscr{D}'(S_T)$-$\mathscr{D}(S_T)$ duality. It remains to be shown that $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ is a distributional derivative of a function ϕ described in the statement.

Let $\alpha \in \mathscr{D}(0,T)$ and $\beta \in \mathscr{D}(\mathbb{R}^n)$ (not necessarily radial) with $\int \beta = 1$. Then

$$\begin{aligned}
\langle \tilde{\phi}_k, \alpha \rangle &= \langle \partial_k\pi_h, \alpha(t)\beta(x) \rangle \\
&= \int\int_{S_T} u_k(y,s)\alpha'(s)\beta(y)\, dy\, ds + \int\int_{S_T} u_k(y,s)\alpha(s)\Delta\beta(y)\, dy\, ds \\
&\quad + \int\int_{S_T} u_j(y,s)u_k(y,s)\alpha(s)\partial_j\beta(y)\, dy\, ds \\
&\quad + \int\int_{S_T} \pi_0(y,s)\alpha(s)\partial_k\beta(y)\, dy\, ds.
\end{aligned}$$

The function we are seeking is therefore

$$\begin{aligned}
\phi_k(t) &= \int_0^t \int u_k(y,s)\Delta\beta(y)\, dy\, ds + \int_0^t \int u_j(y,s)u_k(y,s)\partial_j\beta(y)\, dy\, ds \\
&\quad + \int_0^t \int \pi_0(y,s)\partial_k\beta(y)\, dy\, ds - \int u_k(y,t)\beta(y)\, dy \\
&\quad + \int u_{0k}(y)\beta(y)\, dy
\end{aligned} \tag{3.9}$$

for $k = 1, \dots, n$ and for any fixed $\beta \in \mathscr{D}(\mathbb{R}^n)$ with $\int \beta = 1$. $\square$

Lemma 3.4. *Assume that u is a solution of the initial value problem on $[0, T)$. Then there exists a function*

$$\phi = (\phi_1, \dots, \phi_n) \in L^\infty_{\mathrm{loc}}([0, T), \mathbb{R}^n)$$

with $\lim_{t\to 0+} \phi(t) = \phi(0) = 0$ *such that*

$$u(x, t) = \tilde{u}(x - \Phi(t), t) + \phi(t), \quad (x, t) \in S_T,$$

where $\Phi(t) = \int_0^t \phi(s)\, ds$ and where $\tilde{u}$ is a mild solution on $[0, T)$. A solution of the initial value problem is a mild solution if and only if the associated pressure π satisfies $\pi = K_{ij}(u_i u_j)$.

Proof. Assume that u is a solution of the initial value problem on $[0, T)$. Let $\phi = (\phi_1, \dots, \phi_n)$ be as in the statement of Lemma 3.3; in particular, (3.7) holds in $\mathscr{D}'(S_T)$ with $\pi_0 = K_{ij}(u_i u_j)$. Let $\Phi(t) = \int_0^t \phi(\tau)\, d\tau$. Consider

$$\tilde{u}(x, t) = u(x - \Phi(t), t) + \phi(t),$$

which is also a solution of the initial value problem with the same initial datum and with the associated pressure

$$\tilde{\pi}(x, t) = \pi_0(x - \Phi(t), t)$$

as it can be readily checked (we need $\phi = o(1)$ and $\Phi = o(t^{1/2})$ as $t \to 0$). We now proceed by showing that $\tilde{u}$ is a mild solution. This part follows [FJR] rather closely except that we work with the heat instead of the Oseen kernel. Let $\theta_R(y) = \theta(y/R)$ for $R > 0$ and $y \in \mathbb{R}^n$, where $\theta \in \mathscr{D}(\mathbb{R}^n)$ is identically 1 in a neighborhood of 0. Let $\psi \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \psi \leqslant 1$, $\psi' \geqslant 0$ with $\psi(s) = 0$ for $s \leqslant 1$ and $\psi(s) = 1$ for $s \geqslant 2$. For every fixed $(x, t) \in S_T$ and $\varepsilon > 0$, let

$$\phi(y, s) = \psi\left(\frac{s}{\varepsilon}\right)\psi\left(\frac{t-s}{\varepsilon}\right)\theta_R(y) G(x - y, t - s), \quad (y, s) \in S_T$$

for any $\varepsilon > 0$. Using ϕ as a test function and sending $R \to \infty$, we get

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int\int_{S_T} \tilde{u}_k(y, s)\psi\left(\frac{s}{\varepsilon}\right)\psi'\left(\frac{t-s}{\varepsilon}\right) G(x - y, t - s)\, dy\, ds \\
&\quad + \frac{1}{\varepsilon}\int\int_{S_T} \tilde{u}_k(y, s)\psi'\left(\frac{s}{\varepsilon}\right)\psi\left(\frac{t-s}{\varepsilon}\right) G(x - y, t - s)\, dy\, ds \\
&\quad - \int\int_{S_T} \tilde{u}_k(y, s)\tilde{u}_j(y, s)\psi\left(\frac{s}{\varepsilon}\right)\psi\left(\frac{t-s}{\varepsilon}\right)\partial_j G(x - y, t - s)\, dy\, ds \\
&\quad - \int\int_{S_T} \tilde{\pi}_0(y, s)\psi\left(\frac{s}{\varepsilon}\right)\psi\left(\frac{t-s}{\varepsilon}\right)\partial_k G(x - y, t - s)\, dy\, ds = 0. \qquad (3.10)
\end{aligned}$$

If $0<\varepsilon\leqslant t/4$, the second term on the left-hand side equals

$$\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\int_{\mathbb{R}^n}\tilde{u}_k(y,s)\psi'\left(\frac{s}{\varepsilon}\right)G(x-y,t-s)\,dy\,ds$$

and this expression may be rewritten as

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\psi'\left(\frac{s}{\varepsilon}\right)ds\int\left(\tilde{u}_k(y,s)-\int G(y-w,s)u_{0k}(w)\,dw\right)\\ \times G(x-y,t-s)\,dy+\int G(x-y,t)u_{0k}(y)\,dy.\end{aligned}\tag{3.11}$$

The first term in this expression converges to 0 uniformly in x as $\varepsilon\to 0+$; therefore, as $\varepsilon\to 0+$, equality (3.10) becomes

$$\begin{aligned}&\tilde{u}_k(x,t)-\int\tilde{u}_{0k}(y)G(x-y,t)\,dy\\&\quad+\int_0^t\int\tilde{u}_j(y,s)\tilde{u}_k(y,s)\partial_jG(x-y,t-s)\,dy\,ds\\&\quad+\int_0^t\int\tilde{\pi}_0(y,s)\partial_kG(x-y,t-s)\,dy\,ds=0\end{aligned}$$

for $k=1,\dots,n$ for almost every $(x,t)\in S_T$. Since $\tilde{\pi}_0=K_{ij}(u_iu_j)$, we get that $\tilde{u}$ is a mild solution. The last claim follows by using uniqueness of mild solutions. $\square$

Proof of Theorem 2.2. First, by Proposition 3.2, there exists a (unique) mild solution u on $(0,T)$. The function u is then a solution of the initial value problem by Lemma 3.1.

Now, let $\tilde{u}$ be a solution of the initial value problem. Then, by Lemma 3.4, there exists a mild solution $\bar{u}$ such that

$$\tilde{u}(x,t)=\bar{u}(x-\Phi(t),t)+\phi(t),\quad (x,t)\in S_T,$$

where ϕ and Φ are as in the statement of Lemma 3.4. By uniqueness of mild solutions, $\bar{u}=u$, and the proof is complete. $\square$

Remark 3.5. This remark addresses substitution of (iii) by (iii)$'$ in the definition of solutions of the initial value problem. In this case, all the statements remain precisely the same. There are only two places where (iii) was used. The first instance was at the end of the proof of Lemma 3.3. Assuming (iii), we concluded from (3.9) that $\lim_{s\to 0+}\phi(s)=0$. However, this is clearly the case if (iii)$'$ is assumed instead. Namely, the sum of the last two terms on the right-hand side of (3.9) converges to 0 by (iii)$'$, while the same can be shown for the (only non-trivial) third term by an easy computation. The second instance of use of (iii) was when we showed that the first term in (3.11) converges to 0. This term can be rewritten as

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\psi'\left(\frac{s}{\varepsilon}\right)ds\int\left(\tilde{u}_k(y,s)-\int G(y-w,s)u_{0k}(w)\,dw\right)\theta_R(y)G(x-y,t)\,dy\\
&\quad+\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\psi'\left(\frac{s}{\varepsilon}\right)ds\int\left(\tilde{u}_k(y,s)-\int G(y-w,s)u_{0k}(w)\,dw\right)(1-\theta_R(y))G(x-y,t)\,dy\\
&\quad+\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\psi'\left(\frac{s}{\varepsilon}\right)ds\int\left(\tilde{u}_k(y,s)-\int G(y-w,s)u_{0k}(w)\,dw\right)\\
&\quad\times(G(x-y,t-s)-G(x-y,t))\,dy
\end{aligned}$$

for every $R>0$ where θ_R is as in the proof of Lemma 3.4. This second term can be made arbitrarily small if R is sufficiently large provided x belongs to a bounded set. For any fixed $R>0$, the first and the third term converge to 0 as $\varepsilon\to 0$ in $L^\infty(\mathbb{R}^n)$.

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References

[CF] P. Constantin, C. Foias, Navier–Stokes Equations, Chicago Lectures in Mathematics, University of Chicago, Chicago/London, 1988.

[F] C. Fefferman, Characterizations of bounded mean oscillation, Bull. Amer. Math. Soc. 77 (1971) 587–588.

[FJR] E.B. Fabes, B.F. Jones, N.M. Riviere, The initial value problem for the Navier–Stokes equations with data in L^p, Arch. Rational Mech. Anal. 45 (1972) 222–240.

[FMRT] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes Equations and Turbulence, Encyclopedia of Mathematics and its Applications, Vol. 83, Cambridge University Press, Cambridge, 2001.

[GK] Z. Grujić, I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L^p, J. Funct. Anal. 152 (1998) 447–466.

[K] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, Math. Z. 187 (1984) 471–480.

[KT] H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, Adv. Math. 157 (2001) 22–35.

[S] C. Sadosky, Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis, Marcel Dekker, Inc., New York and Basel, 1979.

[St1] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.

[St2] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.

[T] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, 3rd Revised Edition, North-Holland, Amsterdam, 1984.

[W] F.B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math. 38 (1981) 29–40.